MULTISCALE SPECTRAL VISCOSITY DISCRETIZATION METHODS FOR THE NAVIER-STOKES EQUATIONS

or another explicit filtering turbulence model

or another tweak of Smagorinsky

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FOUR QUOTES RELEVANT TO TURBULENCE MODELING

With four parameters I can fit an elephant,
and with five I can make him wiggle his trunk

attributed by Enrico Fermi
to John von Neumann

All models are wrong, but some models are useful

George Box

Experimental results are believed by everyone
except by those that did the experiment

Computational results are believed by no one
except by those that did the computation

original source unknown

When I die, I will ask my Maker to elucidate me on two things: quantum mechanics and turbulence

I expect satisfaction on quantum mechanics

paraphrasing Augustus Love

LET'S START FROM SCRATCH

- In a remarkable paper published when he was 26 years old, Stokes stated the essential hypotheses about what makes a fluid a fluid
  - The stress tensor $\mathbb{T}$ is a continuous function of the deformation tensor $\mathbb{D} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and is independent of all other kinematic quantities
  - $\mathbb{T}$ does not depend explicitly on the position $\mathbf{x}$ (spatial homogeneity)
  - There is no preferred direction in space (isotropy)
  - When $\mathbb{D} = 0, \quad \mathbb{T} = -p\mathbb{I}$

- It it is shown in

  James Serrin, Mathematical principles of classical fluid mechanics, *Handbuch der Physik* Vol. 8/1, Springer-Verlag, Berlin, 1957

  that the most general constitutive relation satisfying the Stokes hypotheses is given by

$$\mathbb{T} = \alpha\mathbb{I} + \beta\mathbb{D} + \gamma\mathbb{D}^2$$

  where α, β, and γ are scalar function of the principal invariants of $\mathbb{D}$

  - the principal invariants $\mathrm{I}, \mathrm{II}, \mathrm{III}$ of $\mathbb{D}$ are the coefficients in the expansion

$$\det(\xi\mathbb{I} - \mathbb{D}) = \xi^3 - \mathrm{I}\xi^2 + \mathrm{II}\xi - \mathrm{III}$$

    i.e., the characteristic polynomial for $\mathbb{D}$

  - thus $\alpha = \alpha(\mathrm{I}, \mathrm{II}, \mathrm{III})$ and similarly for β and γ

– we have that

$$\mathrm{I} = \operatorname{trace} \mathbb{D} \qquad \mathrm{II} = \tfrac{1}{2}\big(\mathbb{D} : \mathbb{D} - (\operatorname{trace} \mathbb{D})^2\big) \qquad \mathrm{III} = \det \mathbb{D}$$

– thus, $\alpha = \alpha(\operatorname{trace} \mathbb{D}\,,\ \mathbb{D} : \mathbb{D}\,,\ \det \mathbb{D})$ and similarly for β and γ

– note that

- $\operatorname{trace} \mathbb{D}$ is a sum of linear monomials in the entries of $\mathbb{D}$
- $\mathbb{D} : \mathbb{D}$ is a sum of quadratic monomials in the entries of $\mathbb{D}$
- $\det \mathbb{D}$ is a sum of cubic monomials in the entries of $\mathbb{D}$

so that

- I is a sum of linear monomials in the entries of $\mathbb{D}$
- II is a sum of quadratic monomials in the entries of $\mathbb{D}$
- III is a sum of cubic monomials in the entries of $\mathbb{D}$

– note that $\qquad \operatorname{trace} \mathbb{D} = \nabla \cdot \mathbf{u}$

- Assume linearity
  - then
    $$\gamma = 0 \qquad \beta = \beta_1 \qquad \alpha = \alpha_1 + \alpha_2 \mathrm{trace}\,\mathbb{D} = \alpha_1 + \alpha_2 \nabla \cdot \mathbf{u}$$
    where $\alpha_1, \alpha_2, \beta_1$ are constants
  - then $\quad \mathbb{T} = (\alpha_1 + \alpha_2 \nabla \cdot \mathbf{u})\mathbb{I} + \beta_1 \mathbb{D}$
  - by the last Stokes hypothesis, we must have $\quad \alpha_1 = -p$
  - let $\quad \alpha_2 = \lambda \quad$ and $\quad \beta_1 = 2\mu$
  - then
    $$\mathbb{T} = (-p + \lambda \nabla \cdot \mathbf{u})\mathbb{I} + 2\mu \mathbb{D}$$
    is the most general linear constitutive relation satisfying the Stokes hypotheses

- Assume linearity and incompressibility ($\nabla \cdot \mathbf{u} = 0$)
  - then
$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$$
    is, for incompressible flows, the most general linear constitutive relation satisfying the Stokes hypotheses
  - of course, this leads to the Navier-Stokes system
$$\begin{cases} \rho\left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\right) - \rho \mathbf{f} = \nabla \cdot \mathbb{T} = -\nabla p + 2\mu \nabla \cdot \mathbb{D} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

- Assume the flow is incompressible, but is not necessarily linear

  - because $\quad \mathrm{trace}\,\mathbb{D} = \nabla \cdot \mathbf{u}, \quad$ for incompressible flows $\quad \mathrm{I} = \mathrm{trace}\,\mathbb{D} = 0$

  - thus, for incompressible flows, $\quad \alpha = \alpha(\mathbb{D} : \mathbb{D}\,,\ \det \mathbb{D}) \quad$ and similarly for β and γ

  - then

$$\mathbb{T} = \alpha(\det \mathbb{D}, \mathbb{D} : \mathbb{D})\mathbb{I} + \beta(\det \mathbb{D}, \mathbb{D} : \mathbb{D})\mathbb{D} + \gamma(\det \mathbb{D}, \mathbb{D} : \mathbb{D})\mathbb{D}^2$$

    is, for incompressible flows, the most general constitutive relation satisfying the Stokes hypotheses

  - we already know that if we also assume linearity, we get the Navier-Stokes equations

    - let's make a different assumption

- Assume $\gamma = 0$ $\beta = 2\mu + \mu_1 \sqrt{\mathbb{D} : \mathbb{D}}$ $\alpha = -p$

  – this leads to the system

$$\begin{cases} \rho \left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \rho \mathbf{f} = \nabla \cdot \mathbb{T} \\ \qquad\qquad = -\nabla p + 2\mu \nabla \cdot \mathbb{D} + \mu_1 \nabla \cdot \left((\mathbb{D} : \mathbb{D})^{1/2} \mathbb{D} \right) \\ \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

  – looks like Smagorinsky (1963) to me

- The clever modeler can come up with all sorts of other choices for α, β, and γ

  – for example, some have advocated a polynomial dependence of $\mathbb{T}$ on the entries of $\mathbb{D}$

A MILLION DOLLAR PROBLEM

- The existence of weak solutions of the Navier-Stokes system was proved in the 1930's by Jean Leray

- In two dimensions, the uniqueness of weak solutions has also been established

- However, in three dimensions, the global uniqueness of weak solutions has not been proved
  - uniqueness has been proved only for smooth solutions

- Closing this gap gets you $1,000,000 from the Clay Institute

WHY IS IT DIFFICULT TO SOLVE THE $1,000,000 PROBLEM?

- Using dimensional analyses, Terence Tao gave a heuristic indication of why it is difficult to show the well posedeness of the Navier-Stokes equations

- Suppose we set $\mathbf{f} = \mathbf{0}$ and assume that $\mathbf{u}$ has frequency support of order N and hence has a physical support of order N^{-1}

- Then, the energy dissipation estimate tells us that the following global quantities are controlled:

$$\|\mathbf{u}\| = \int_{\mathbb{T}^d} |\mathbf{u}|^2 \sim U^2 N^{-d} \qquad \text{and} \qquad \int_0^T \int_{\mathbb{T}^d} |\nabla \mathbf{u}|^2 \sim T N^2 U^2 N^{-d}$$

  – here, d denotes the space dimension

  – we assume periodicity

- Due to the skew-symmetry of the nonlinear term, its effect on the globally conserved quantities is nil
  - however, it can have a local effect of order

$$\int_0^T \int_{Local} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \sim T U^3 N N^{-d}$$

- Assuming that $\|\mathbf{u}(0)\| \sim O(1)$, we have

$$\|\mathbf{u}(t)\|^2 \sim U^2 N^{-d} \sim O(1) \qquad \text{so that} \qquad U \sim N^{d/2}$$

- It follows that the dissipative effect is of order

$$\int_0^T \int_{\mathbb{T}^d} |\nabla \mathbf{u}|^2 \sim T N^2$$

wheras the nonlinear convective effect is of order

$$\int_0^T \int_{Local} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \sim T N^{1+d/2}$$

- For $d = 2$ the linear term and the nonlinear term have the same dimension
  - hence, we say that the energy dissipation bound is critical for the 2D Navier-Stokes equations

- For $d = 3$, because $1 + \frac{d}{2} > 2$, the nonlinear term dominates and hence we call the energy bound supercritical
  - the problem in 3D is that the global energy bound is supercritical so that it does not provide enough control on the size of $\mathbf{u}$ to prevent local instabilities due to the nonlinearity from happening

- Note that energy dissipation also says that $TN^2 \sim O(1)$ or $T \sim N^{-2}$, which is the well-known parabolic scaling $T \sim L^2$

- This roughly means that the solution can have the Nth mode staying large only for a time interval of length N^{-2}

- This implies that the nonlinearity is of the order $TN^{1+d/2} \sim N^{d/2-1}$
  - this is a positive power of N for $d \geq 3$

- Hence, it leaves the possibility for energy to be cascaded to higher and higher frequencies, only staying within any one frequency in a time interval of duration N^{-2}, and meanwhile increasing the effect of the nonlinearity

- But the sum of N^{-2} as $N \to \infty$ is finite
  - therefore, the solution can blow up in finite time due to ever increasing nonlinear destabilizing effect
  - however, no such blowup is known to happen and hence the $1,000,000 problem remains open

OLGA LADYZHENSKAYA ENTERS THE PICTURE

- At the 1966 International Congress of Mathematicians, Olga Ladyzhenskaya proposed several modifications of the Navier-Stokes equations
  - her modifications started with what Stokes' hypotheses allows one to do
  - one of her modifications:

$$\gamma = 0 \quad \text{and} \quad \beta = 2\mu + \mu_1 \left((\mathbb{D} : \mathbb{D})^{1/2} \right)^{q-2}$$

  - Smagorinsky is the special case $q = 3$
  - we then have the modified Navier-Stokes system

$$\begin{cases} \rho \left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \rho \mathbf{f} = -\nabla q + 2\mu \nabla \cdot \mathbb{D} + \mu_1 \nabla \cdot \left((\mathbb{D} : \mathbb{D})^{\frac{q-2}{2}} \mathbb{D} \right) \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

- Ladyzhenskaya's interests were about proving well posedness
  - she (as well as everyone else) could not and still cannot prove it for the 3D Navier-Stokes equations
  - but, she was able to prove well posedness for the modified Navier-Stokes equations for sufficiently large q
  - the game since then has been to try to reduce the value of q for which well posedness can be proved
    - the "best" result to date is that due to Malek et al. who proved the existence and global uniqueness of a strong solution for $q \geq 11/5$
    - note that this means that the Smagorinsky model ($q = 3$) is well posed
    - in fact, Ladyzhenskaya's proof already included the Smagorinsky model as a special case

HYPER-VISCOSITY REGULARIZATION

- Another model that attains well-posedness is the hyper-viscosity model

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} - \epsilon(-\Delta)^s \mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

  – the fractional Laplacian operator is defined via the Fourier transform:

$$\widehat{(-\Delta)^s v}(k) = |k|^{2s} \widehat{v}(k)$$

  – J.-L. Lions showed that unique strong solution exist for $s > 5/4$

WHAT DOES TERENCE TAO HAVE TO SAY ABOUT THE MODIFIED MODELS?

- We apply Tao's reasoning to the Ladyzhenkaya and hyper-viscosity models

- For the Ladyzhenkaya model, we have

$$\int_0^T \int_{\mathbb{T}^d} |\nabla \mathbf{u}|^q \sim T U^q N^q N^{-d} \sim T N^{d(\frac{q}{2}-1)+q}$$

which in 3D becomes critical at $p = 11/5$

- For the hyper-viscosity model, we have

$$\int_0^T \int_{\mathbb{T}^d} ||\nabla|^s \mathbf{u}|^2 \sim T U^2 N^{2s} N^{-d} \sim T N^{2s}$$

which in 3D becomes critical at $s = 5/4$

- Note how simple dimensional analysis is a powerful tool in predicting the criticality of the models, and therefore provides foresight into the well-posedness question

SPECTRAL VISCOSITY LADYZHENSKAYA MODELS

- In order to preserve large-scale structures, we would like to limit the regularization effect to the smaller scales

- In fact, this is essentially the idea of the spectral viscosity method for hyperbolic conservation laws introduced by Etan Tadmor and co-workers
  - viscosity is added only at high frequencies
  - in this way, it is hoped that stabilization is effected without hurting the large-scale features of the solution
  - in the hyperbolic conservation law case, this is exactly what happens

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Joint work with Yuki Saka and Xiaoming Wang (Florida State University), Catalin Trenchea (University of Pittsburgh), and Eunjung Lee (Yonsei University, Korea)

- Inspired by the spectral viscosity method for hyperbolic equations, we could consider the following spectral viscosity model for Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} - \epsilon Q(\nabla \cdot (Q(\nabla \mathbf{u})) + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

  where Q is a high-pass filter

  - the action of Q erases all low-frequency modes of the input
  - adding such a linear filtered viscosity term to hyperbolic conservation laws makes the resulting modified problem well posed
  - this is not the case for the Navier-Stokes equations
  - so, instead, we apply the filtered viscosity idea to the Ladyzhenskaya model

- We modify the Ladyzhenskaya model so that the regularization only affects the high-frequency modes:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} - \epsilon Q \nabla \cdot \left(|\nabla Q \mathbf{u}|^{q-2} \nabla Q u \right) + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

  – this is the model we study

- A model similar to this (with $q = 3$, the Smagorinsky q) was used by Jansen and Tejada-Martinez for a numerical simulation
  - a two-grid finite element method for which the cut-off operator is defined by an appropriate orthogonal projection onto the fine scales
  - their numerical simulations showed how the spectral eddy viscosity model leaves larger scale structures alone

- Karamanos and Karniadakis implemented the linear version of the spectral viscosity method
  - this model is not known to be well posed
  - of course, they do not have any proofs

- In addition to q, there are two parameters that have to be specified:

  - the coefficient ϵ of the modification term

  - the cut-off frequency M of the high-pass filter Q

  – large ϵ and low M results in a more stable model at the expense of increasing the modeling error due to the over-damping the low frequency modes

  – small ϵ and high M increases consistency at the expense of stability

- We are one short of the 4 parameters von Neumann said he needed

  - we certainly can't make the elephant's trunk wiggle

SPECTRAL HYPER-VISCOSITY MODELS

- In a similar manner we can define a spectral hyper-viscosity model by

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \epsilon(-\Delta)^s Q_M \mathbf{u} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

  – this model is also determined by three parameters M, s, and ϵ

WELL-POSEDNESS OF THE SPECTRAL VISCOSITY MODELS

- We have proven the well posedness of the spectral-viscosity/Ladyzhenskaya model for $q \geq 11/5$

- We have studied the convergence to solutions of the Navier-Stokes equations as $\epsilon \to 0$ and $M \to \infty$
  - the M has to depend on ϵ
  - an optimal M also depends on q

- We have also proved the convergence of Fourier approximations

- We have applied Terence Tao's arguments to the filtered model to show that again there is enough dissipation to stabilize the system

- Many results similar to the ones we proved for the spectral-viscosity/Ladyzhenskaya model were proven by Guermond and Prudhomme or can also be proven (more easily) for the spectral-viscosity/hyperviscosity model

IMPLEMENTATION

- Instead of Fourier methods, one can use any other multiscale basis
  - we would like to use hierarchical finite element bases or hierarchical wavelet bases
  - we have already done this for hyperbolic conservation laws using a

    hierarchical finite element bases (with M. Calhoun-Lopez)

    and

    hierarchical wavelet bases (with A. Diez and A. Kunoth)
  - want to do this for the spectral viscosity/Ladyzhenskaya model

PAPER WITH DETAILS ABOUT OUR WORK

- Max Gunzburger, Eunjung Lee, Yuki Saka, Catalin Trenchea, and Xiaoming Wang, *Analysis of nonlinear spectral eddy-viscosity models of turbulence*; J. Scien. Comput. **45** 2010, 294-332.

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